CALCULATION OF THE SELF-OSCILLATIONS ARISING AS A RESULT OF THE LOSS OF STABILITY BY THE SPIRAL FLOW OF A VISCOUS LIQUID IN AN ANNULAR TUBE

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The spiral flow of a viscous liquid in an annular gap (formed by concentric cylinders) due to the rotation of the inner cylinder and the axial pressure gradient is considered; the stability of the flow is discussed in relation to small but finite rotationally symmetrical perturbations.

The theory of the stability of spiral flows is usually considered on a linear basis and considerable simplifying assumptions are made; the perturbations are assumed to have rotational symmetry, the gap between the cylinders is considered as being narrow, and the axial Reynolds number is regarded as small [1-5]. No limitations were imposed on the axial flow in [6], in which the problem was solved by an asymptotic method on the approximation of a narrow gap. An analysis of stability with no limitations imposed upon the width of the gap, based on the equations of an ideal liquid, was presented in [7]; the case of cylinders slipping relative to one another and nonrotationally symmetrical perturbations was considered in [8]. The influence of an axial flow on the stability limits was studied experimentally in [9-14]; self-oscillations in an annular tube were observed in [11, 12].

A detailed numerical study of the stability of spiral flows was presented in [15, 16]; in addition to the rotationally-symmetrical case, three-dimensional oscillations were considered, and the neutral curves were calculated over a wide range of variation of Reynolds numbers, gap widths, and longitudinal wave numbers. A strict proof of the existence of a situation, periodic in time, arising as a result of the loss of stability of the spiral flow due to rotation and a very slow translational motion of the cylinder was presented in [17]; one example of the generation of convective self-oscillations of the flow of a viscous liquid in a cylindrical tube was considered in [18].

In this paper we shall use the Lyapunov–Schmidt method [17, 19-21] in considering the case of a narrow channel, in which axisymmetrical perturbations are the most dangerous [16]; we shall calculate the amplitude of the secondary transient laminar mode and study its stability for various values of the Reynolds number Re_z, constructed from the axial velocity. We shall show that, if the parameter $Re_z < 40$ and the Reynolds number of the rotational component of the spiral flow exceeds the critical value given by the linear theory, soft excitation of a stable, self-oscillatory flow having the form of waves traveling in the liquid along the axis of the cylinders will occur. The main spiral flow will then lose stability.

1. Presentation of the Problem

Let a viscous, incompressible liquid of density ρ with a kinematic viscosity ν occupy the space between two concentric cylinders of radii r_1 and r_2 ($r_1 < r_2$). The inner cylinder rotates uniformly with an angular velocity Γ, the outer one remains stationary. We take r_2-r_1, $(r_2-r_1)^2/\nu$, $\rho(r_2-r_1)^3$ as units of length, time, and mass and introduce a cylindrical coordinate system r, θ, z', in which the z' axis coincides with the axis of the cylinders. As we are interested in time-periodic modes of flow possessing rotational symmetry $(\partial/\partial\theta \equiv 0)$ and a specified periodicity along the z' axis, we shall seek solutions to the dimensionless hydrodynamic equations in the Gromeko–Lamb form

$$\partial \mathbf{v}'/\partial t + \boldsymbol{\omega}' \times \mathbf{v}' + \operatorname{rot} \boldsymbol{\omega}' + \operatorname{grad} h' = 0,\ \boldsymbol{\omega}' = \operatorname{rot} \mathbf{v}',\ \operatorname{div} \mathbf{v}' = 0, \quad (1.1)$$

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solely dependent upon the variables r and $z = z' - c't$ (c' is the unknown phase velocity of the waves). It was verified in [17] that in the axisymmetrical case every isolated limiting cycle of the system (1.1) admitted a representation of this kind, in view of the invariance of the equations in relation to shear $z' \to z' + \text{const}$. The solutions which we are seeking should be $2\pi/\alpha$ periodic with respect to z (α is the specified wave number). We see from the equations of motion that the total pressure gradient, averaged over the period, does not depend on the transverse coordinate r, and we assume it as prespecified:

$$\frac{\alpha}{2\pi}\int_{-\pi/\alpha}^{\pi/\alpha}\frac{\partial h'}{\partial z}dz = \text{const}.$$

The foregoing conditions are satisfied by the z-independent steady-state solution (spiral flow)

$$\mathbf{v}' = \mathbf{V}(r) = (0,\ V_\theta,\ V_z);\ \boldsymbol{\omega}' = \boldsymbol{\Omega}(r) = (0,\ \Omega_\theta,\ \Omega_z) = \operatorname{rot}\mathbf{V};$$
$$h' = H = \text{const}\, z + 0{,}5\left(V_\theta^2 + V_z^2\right) + \int V_\theta^2/r\,dr \tag{1.2}$$

with velocity components

$$V_\theta = \mathrm{Re}_\theta(d_1/r - d_2 r);\ V_z = \mathrm{Re}_z[d_3(r^2 - \xi^2) - d_4\ln(r/\xi)],$$
$$\xi = r_1/(r_2 - r_1),\ d_2 = \xi/(1 + 2\xi),\ d_1 = (1 + \xi)^2 d_2,$$
$$d_4 = 2\Big/\left[1 - \frac{1 + 2\xi + 2\xi^2}{1 + 2\xi}\ln(1 + 1/\xi)\right],\ d_3 = d_4\frac{\ln(1 + 1/\xi)}{1 + 2\xi},$$

where $\mathrm{Re}_\theta = \Gamma r_1(r_2 - r_1)/\nu$ is the Reynolds number of the azimuthal component V_θ constructed from the maximum velocity Γr_1 and the width of the channel; Re_z is the Reynolds number of the axial flow V_z based on the viscosity ν, the channel width $r_2 - r_1$, and the axial velocity averaged over the cross section of the annular tube, so that we have the equation

$$2\pi\int_{\xi}^{1+\xi} V_z r\,dr = \pi\left[(1 + \xi)^2 - \xi^2\right]\mathrm{Re}_z,$$

in the case of a narrow gap ($\xi \to \infty$) the profile V_z passes into parabolic form $V_z = 6\,\mathrm{Re}_z y(1-y)$, $y = r - \xi$.

Seeking the periodic modes branching from solution (1.2), let us substitute the following into Eq. (1.1):

$$\mathbf{v}' = \mathbf{V} + \mathbf{v}(r, z),\ \boldsymbol{\omega}' = \boldsymbol{\Omega} + \boldsymbol{\omega}(r, z),\ h' = H + h(r, z),\ c' = c\,\mathrm{Re}_z;$$

in order to determine the perturbations of $\mathbf{v}$, h, $\boldsymbol{\omega}$ and the constant c we obtain a nonlinear problem with respect to the eigenvalues

$$-c\,\mathrm{Re}_z d\mathbf{v}/dz + \boldsymbol{\Omega}\times\mathbf{v} + \boldsymbol{\omega}\times\mathbf{V} + \operatorname{rot}\boldsymbol{\omega} + \operatorname{grad} h = \mathbf{v}\times\boldsymbol{\omega},\ \boldsymbol{\omega} = \operatorname{rot}\mathbf{v},$$
$$\operatorname{div}\mathbf{v} = 0,\ \int_{-\pi/\alpha}^{\pi/\alpha}\frac{\partial h}{\partial z}dz = 0,\ \mathbf{v} = 0\,(r = \xi,\ 1 + \xi), \tag{1.3}$$

for which it is required to find a nonzero solutions $2\pi/\alpha$ periodic in the coordinate z.

2. Lyapunov – Schmidt Series

Let Re_0 be the critical value of the parameter Re_θ. Let us put $\mathrm{Re}_\theta = \mathrm{Re}_0 + \varepsilon^2$, and regarding ε as small, seek the solution to the problem (1.3) in the form of [19]:

$$(\mathbf{v},\ h,\ \boldsymbol{\omega}) = \sum_{k=1}^{\infty}\varepsilon^k(\mathbf{v}_k,\ h_k,\ \boldsymbol{\omega}_k),\ c = \sum_{k=0}^{\infty}\varepsilon^k c_k \tag{2.1}$$

We then arrive at a series of recurrence problems (k=1, 2, 3...)

$$-c_0\,\mathrm{Re}_z\,\partial\mathbf{v}_k/dz + \boldsymbol{\Omega}_0\times\mathbf{v}_k + \boldsymbol{\omega}_k\times\mathbf{V}_0 + \operatorname{rot}\boldsymbol{\omega}_k + \operatorname{grad} h_k = \mathbf{f}_k,$$
$$\operatorname{rot}\mathbf{v}_k = \boldsymbol{\omega}_k,\ \operatorname{div}\mathbf{v}_k = 0,\ \mathbf{v}_k = 0\ (r = \xi,\ 1 + \xi), \tag{2.2}$$
$$\int_{-\pi/\alpha}^{\pi/\alpha}\frac{\partial h_k}{\partial z}dz = 0,\ (\mathbf{v}_k,\ h_k,\ \boldsymbol{\omega}_k)|_z \equiv (\mathbf{v}_k,\ h_k,\ \boldsymbol{\omega}_k)|_{z+2\pi/\alpha}$$

with known right-hand sides. For example,

$$\mathbf{f}_1 = 0,\ \mathbf{f}_2 = \mathbf{v}_1 \times \boldsymbol{\omega}_1 + c_1 \operatorname{Re}_z \partial \mathbf{v}_1/\partial z,$$

$$\mathbf{f}_3 = \mathbf{v}_1 \times \boldsymbol{\omega}_2 + \mathbf{v}_2 \times \boldsymbol{\omega}_1 + c_2 \operatorname{Re}_z \partial \mathbf{v}_1/\partial z + c_1 \operatorname{Re}_z \partial \mathbf{v}_2/\partial z - (\boldsymbol{\Omega}_1 \times \mathbf{v}_1 + \boldsymbol{\omega}_1 \times \mathbf{V}_1).$$

Here we have introduced the notation

$$\mathbf{V}_0 = \mathbf{V}|_{\operatorname{Re}_\theta = \operatorname{Re}_0},\ \mathbf{V}_1 = \mathbf{V}|_{\substack{\operatorname{Re}_\theta = 1 \\ \operatorname{Re}_z = 0}},\ \boldsymbol{\Omega}_0 = \operatorname{rot} \mathbf{V}_0,\ \boldsymbol{\Omega}_1 = \operatorname{rot} \mathbf{V}_1.$$

When k=1 we obtain a linear homogeneous problem for calculating the critical parameters Re_0, c_0 and the eigenfunction. The solution to this problem may be sought in the form

$$(\mathbf{v}_1, h_1, \boldsymbol{\omega}_1) = \beta(\boldsymbol{\Phi}(y)e^{i\alpha z} + \boldsymbol{\Phi}^*(y)e^{-i\alpha z}),$$
$$\boldsymbol{\Phi} = (\boldsymbol{\varphi}, g, \boldsymbol{\gamma}),\ y = r - \xi, \tag{2.3}$$

in which the unknown real constant β (amplitude of the self-oscillations) may be regarded as positive, since this state of affairs may always be achieved by moving the origin along the z axis. The asterisk denotes the operation of complex conjugation. Separating the variable z, we arrive at a system of six differential equations of the first order:

$$\begin{gathered} D\varphi_r = -q\varphi_r - i\alpha\varphi_z;\ D\varphi_\theta = \gamma_z - q\varphi_\theta;\ D\varphi_z = i\alpha\varphi_r - \gamma_\theta; \\ Dg = i\alpha c_0 \operatorname{Re}_z \varphi_r + i\alpha\gamma_\theta - A_r;\ D\gamma_\theta = i\alpha c_0 \operatorname{Re}_z \varphi_z - i\alpha h - q\gamma_\theta - A_z; \\ D\gamma_z = i\alpha\gamma_r - i\alpha c_0 \operatorname{Re}_z \varphi_\theta + A_\theta,\ \gamma_r = -i\alpha\varphi_\theta,\ q = (\xi + y)^{-1}, \\ D = d/dy,\ \mathbf{A} = \boldsymbol{\Omega}_0 \times \boldsymbol{\varphi} + \boldsymbol{\gamma} \times \mathbf{V}_0. \end{gathered} \tag{2.4}$$

for which it is required to seek a nonzero solution satisfying the boundary conditions $\varphi_r = \varphi_\theta = \varphi_z = 0$ (y = 0, 1). By way of normalization it is convenient to take the condition $\gamma_z = 1$ at y = 0. For this choice of normalization the quantity $2\beta\varepsilon$ may in the case of small ε be interpreted as the amplitude of the pulsations of the tangential stress $p_{r\theta}$ on the inner cylinder.

In order to construct the conjugate problem [19] we scalar-multiply the first equation of (2.2) for k=1 by the solenoidal $2\pi/\alpha$ periodic (in z) vector $\boldsymbol{\Psi}$, which vanishes at $r = \xi$, $1 + \xi$, and integrate over the rectangle $\{\xi \le r \le 1 + \xi,\ -\pi/\alpha \le z \le \pi/\alpha\}$ with weight r. If we then integrate by parts, change the derivatives in $\mathbf{v}_1$, h_1, $\boldsymbol{\omega}_1$ to $\boldsymbol{\Psi}$, and introduce the auxiliary variables P and $\boldsymbol{\Lambda}$, we arrive at the conjugate problem

$$c_0 \operatorname{Re}_z \partial \boldsymbol{\Psi}/\partial z + \boldsymbol{\Psi} \times \boldsymbol{\Omega}_0 + \operatorname{grad} P + \operatorname{rot} \boldsymbol{\Lambda} = 0,\ \operatorname{div} \boldsymbol{\Psi} = 0,$$
$$\operatorname{rot} \boldsymbol{\Psi} + \mathbf{V}_0 \times \boldsymbol{\Psi} = \boldsymbol{\Lambda},\ \boldsymbol{\Psi} = 0\ (r = \xi,\ 1 + \xi).$$

which after separation of the variable $z(\boldsymbol{\Psi}, P, \boldsymbol{\Lambda}) = (\boldsymbol{\psi}, p, \boldsymbol{\lambda})e^{i\alpha z}$ reduces to the system

$$\begin{gathered} D\psi_r = -i\alpha\psi_z - q\psi_r;\ D\psi_\theta = \lambda_z - q\psi_\theta + V_{0\theta}\psi_r; \\ D\psi_z = i\alpha\psi_r - \lambda_\theta + V_{0z}\psi_r;\ Dp = i\alpha\lambda_\theta - i\alpha c_0 \operatorname{Re}_z \psi_r + \Omega_{0\theta}\psi_z - \Omega_{0z}\psi_\theta; \\ D\lambda_\theta = -i\alpha p - q\lambda_\theta - i\alpha c_0 \operatorname{Re}_z \psi_z - \Omega_{0\theta}\psi_r; \\ D\lambda_z = i\alpha\lambda_r + i\alpha c_0 \operatorname{Re}_z \psi_\theta - \Omega_{0z}\psi_r,\ \lambda_z = V_{0\theta}\psi_z - (i\alpha + V_{0z})\psi_\theta \end{gathered} \tag{2.5}$$

with boundary conditions $\psi_r = \psi_\theta = \psi_z = 0$ (y = 0, 1) and the additional normalization condition $\lambda_z = 1$ at y = 0. The condition for the solubility of the inhomogeneous problem (2.2) takes the form

$$\int_{-\pi/\alpha}^{\pi/\alpha} \int_0^1 (\mathbf{f}_k(y, z), \boldsymbol{\psi}(y))\, e^{-i\alpha z} r\, dy\, dz = 0\ (k = 2, 3, 4, \ldots). \tag{2.6}$$

Applying this to the case of k=2 and considering that by virtue of (2.3)

$$\mathbf{f}_2 = i\alpha c_1 \operatorname{Re}_z (e^{i\alpha z}\boldsymbol{\varphi} - e^{-i\alpha z}\boldsymbol{\varphi}^*) + \beta^2(\boldsymbol{\varphi} \times \boldsymbol{\gamma}^* + \boldsymbol{\varphi}^* \times \boldsymbol{\gamma} + e^{2i\alpha z}\boldsymbol{\varphi} \times \boldsymbol{\gamma} + e^{-2i\alpha z}\boldsymbol{\varphi}^* \times \boldsymbol{\gamma}^*), \tag{2.7}$$

we find that $c_1 = 0$ if

$$I_1 = \int_0^1 (\boldsymbol{\varphi}, \boldsymbol{\psi})\, r\, dy$$

is nonzero. The latter condition was verified numerically, and it was found to be satisfied in the cases under consideration. The solution of problem (2.2) for k=2 may, in accordance with (2.7), be sought in the form

$$(\mathbf{v}_2, h_2, \boldsymbol{\omega}_2) = \beta^2[(\mathbf{W}, S, \mathbf{L}) + (\mathbf{w}, s, \mathbf{l})\, e^{2i\alpha z} + (\mathbf{w}^*, s^*, \mathbf{l}^*)e^{-2i\alpha z}]. \tag{2.8}$$

TABLE 1

Re_z	α	Re_0	c_0	β	$10^2 c_2$	$10^2\,\mathrm{Real}\,\sigma_2$
4,30	3.13	296.60	1.1696	19.58	−3.33	8,94
11,3	3.14	308.82	1.1683	19,73	−2.96	9,19
17,8	3,15	329,31	1,1660	20.02	−1,33	9.61
29,2	3,20	383,62	1.1606	20.78	4.98	10,7
39,7	3.25	450.31	1.1545	21,95	27.5	12.1

We note that the right-hand side of Eq. (2.8) should be supplemented with the solution to the one-dimensional problem Φ, having a certain numerical factor β_1; however, from the condition of solubility for k=4 we find that $\beta_1=0$. Substitution gives the following equations for the coefficient of the zero harmonic:

$$\begin{gathered} DW_\theta = L_z - qW_\theta;\ DW_z = -L_\theta,\ W_\theta = W_z = 0\ (y = 0,\ 1); \\ DL_\theta = B_z - qL_\theta;\ DL_z = -B_\theta;\ W_r = L_r \equiv 0; \\ DS = B_r - \Omega_{0\theta}W_z + \Omega_{0z}W_\theta - V_{0z}L_\theta + V_{0\theta}L_z. \\ \mathbf{B} = \boldsymbol{\varphi}\times\boldsymbol{\gamma}^* + \boldsymbol{\varphi}^*\times\boldsymbol{\gamma},\ S = 0\ (y = 1). \end{gathered} \tag{2.9}$$

The additional boundary condition for S establishes the arbitrary constant in the definition of the total pressure. We seek the coefficients of the second harmonic by solving the boundary-value problem

$$\begin{gathered} Dw_r = -qw_r - 2i\alpha w_z;\ Dw_\theta = l_z - qw_\theta;\ Dw_z = 2i\alpha w_r - l_\theta; \\ Ds = 2i\alpha(c_0\,\mathrm{Re}_z\,w_r + l_\theta) - C_r;\ Dl_\theta = 2i\alpha(c_0\,\mathrm{Re}_z\,w_z - s) - ql_\theta - C_z; \\ Dl_z = 2i\alpha(l_r - c_0\,\mathrm{Re}_z\,w_\theta) + C_\theta,\ w_r = w_\theta = w_z = 0\ (y = 0,\ 1). \\ l_r = -2i\alpha w_\theta,\ \mathbf{C} = \boldsymbol{\Omega}_0\times\mathbf{w} + \mathbf{l}\times\mathbf{V}_0 - \boldsymbol{\varphi}\times\boldsymbol{\gamma}. \end{gathered} \tag{2.10}$$

Using (2.6) with k=3, we obtain the equation

$$i\alpha c_2\,\mathrm{Re}_z I_1 + \beta^2 I_2 = I_3.$$

$$I_2 = \int_0^1 (\boldsymbol{\varphi}\times\mathbf{L} + \mathbf{W}\times\boldsymbol{\gamma} + \boldsymbol{\varphi}^*\times\mathbf{l} - \mathbf{w}\times\boldsymbol{\gamma}^*,\ \boldsymbol{\psi})\,r\,dy.$$

$$I_3 = \int_0^1 (\boldsymbol{\Omega}_1\times\boldsymbol{\varphi} + \boldsymbol{\gamma}\times\mathbf{V}_1,\ \boldsymbol{\psi})\,r\,dy,$$

and on solving this we find the real constants β and c_2:

$$\beta = \sqrt{\mathrm{Real}\,(I_1^* I_3)/\mathrm{Real}\,(I_1 I_2^*)}\ ,\ c_2 = \frac{\mathrm{Im}\,(I_2^* I_3)}{\alpha\,\mathrm{Re}_z\,\mathrm{Real}\,(I_1 I_2^*)}\ .$$

Computer calculations showed that for the values of the parameters under consideration the quantity under the root was positive. This indicates [19, 20] that the series (2.1) converges and for small ε the self-oscillatory solution so constructed, existing in the hypercritical region $\mathrm{Re}_\theta = \mathrm{Re}_0 - \varepsilon^2$, is unique.

In order to study the stability of the spiral flow and the branching wave mode in the class of perturbations with period $2\pi/\alpha$ in z', we use Eq. (1.1) twice to set up equations in variational form, and we seek the velocity vector as $\mathbf{u}(r,\ z' - c_0\,\mathrm{Re}_z\,t)\exp\sigma t$ in the first case and as $\mathbf{u}'(r,\ z' - c't)\exp\sigma' t$ in the second. This leads to the following problems for the eigenvalues of the exponential indices σ and σ'.

$$\left\{\begin{aligned} &\sigma\mathbf{u} - c_0\,\mathrm{Re}_z\,\partial\mathbf{u}/\partial z + \boldsymbol{\Omega}\times\mathbf{u} + \mathrm{rot}\,\mathbf{u}\times\mathbf{V} + \mathrm{grad}\,\chi + \mathrm{rot}\,\mathrm{rot}\,\mathbf{u} = 0, \\ &\mathrm{div}\,\mathbf{u} = 0,\ \mathbf{u} = 0\ (r = \xi,\ 1 + \xi),\ z = z' - c_0\,\mathrm{Re}_z\,t); \end{aligned}\right.$$

$$\left\{\begin{aligned} &\sigma'\mathbf{u}' - c'\partial\mathbf{u}'/\partial z + (\boldsymbol{\Omega} + \boldsymbol{\omega})\times\mathbf{u}' + \mathrm{rot}\,\mathbf{u}'\times(\mathbf{V} + \mathbf{v}) + \mathrm{grad}\,\chi' + \\ &+ \mathrm{rot}\,\mathrm{rot}\,\mathbf{u}' = 0,\ \mathrm{div}\,\mathbf{u}' = 0,\ \mathbf{u}' = 0\ (r = \xi,\ 1 + \xi),\ z = z' - c't. \end{aligned}\right.$$

Since we are interested in the behavior of the perturbation for a slight increment over the critical state, we may seek solutions to the foregoing problems in the form of series in ε [20] and arrive at the result

$$\sigma = \sigma_2\varepsilon^2 + 0(\varepsilon^2);\ \sigma' = \sigma_2'\varepsilon^2 + 0(\varepsilon^2),$$

$$\sigma_2 = -I_3/I_1,\ \mathrm{Real}\,\sigma_2' = -\mathrm{Real}\,\sigma_2,$$

in which the existence of a positive real part in the coefficient σ_2 or σ'_2 indicates the instability of the corresponding flow.

3. Numerical Results

Calculations of the self-oscillatory situation were carried out on the ODRA-1204 computer for a gap of $\xi = 50$ and various Re_z numbers. First we used the Newton method to refine the critical values of the parameters c_0, Re_0 found in [15]; the wave number was chosen from the consideration of reducing the critical Reynolds number $Re_0(\alpha)$ to a minimum. Then we solved the boundary-value problems (2.4), (2.5), (2.9), and (2.10) by a complex version of the orthogonalization method [22, 23]; the calculation of the integrals I_1, I_2, I_3 amounted to obtaining a solution of the Cauchy problem from the outer to the inner cylinder, with the parallel integration of a larger and larger system of differential equations by the standard Runge–Kutta method of the fourth order, with automatic step selection.

For moderate axial Reynolds numbers the results shown in Table 1 enable us to interpret the solution here obtained as the stable wave motion of the liquid due to the removal of the secondary Taylor vortices by the axial flow.

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OSCILLATIONS OF AN IDEAL LIQUID ACTED UPON BY SURFACE-TENSION FORCES. CASE OF A DOUBLY CONNECTED FREE SURFACE

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Many articles have appeared on the problems of small oscillations of an ideal liquid acted upon by surface-tension forces. Oscillations of a liquid with a single free surface are treated in [1, 2]. Oscillations of an arbitrary number of immiscible liquids bounded by equilibrium surfaces on which only zero volume oscillations are assumed possible are investigated in [3]. We consider below the problem of the oscillations of an ideal liquid with two free surfaces on each of which nonzero volume disturbances are kinematically possible. The disturbances satisfy the condition of constant total volume. A method of solution is presented. The problem of axisymmetric oscillations of a liquid sphere in contact with the periphery of a circular opening is considered neglecting gravity. The first two eigenfrequencies and oscillatory modes are found.

§ 1. Suppose a certain volume Q of an ideal liquid bounded by solid walls of a container S and two free surfaces Σ_1 and Σ_2 (Fig. 1) is in a state of stable equilibrium; ρ is the density of the liquid, and σ_1 and σ_2 are the surface tensions. The external field of body forces has the potential Π.

We consider small oscillations of the liquid about the equilibrium position. We denote by $\mathbf{n}_i(\xi)$ the normal to the undisturbed surface Σ_i ($i=1, 2$) at the point ξ directed outward from the region Q, and by $u_i(\xi, t)$ a small displacement along $\mathbf{n}_i$ at time $t \geq 0$. We assume that the displacement $u_i(\xi, t)$ is a twice continuously differentiable function of the parameter $\xi (\in \Sigma_i)$. We denote by D_i the set of such functions. Let $D = D_1 \times D_2$ be the space of all pairs of functions $\{u_1, u_2\}$ where $u_i \in D_i$. We use the vector notation $\mathbf{u} = \{u_1, u_2\}$ for the elements of the set D. We define the scalar product in D ($\mathbf{u}, \mathbf{v} \in D$)

$$(\mathbf{u}, \mathbf{v}) = \int_{\Sigma_1} u_1 v_1 d\Sigma_1 + \int_{\Sigma_2} u_2 v_2 d\Sigma_2.$$

We introduce the displacement potential $\Phi(q, t)$, $q \in Q$ to describe small oscillations of an ideal liquid [4]. For any $t \geq 0$ the potential Φ is a solution of the problem

$$\Delta\Phi = 0, \quad q \in Q;$$
$$\partial\Phi/\partial\mathbf{n}|_S = 0; \quad \partial\Phi/\partial\mathbf{n}_i|_{\Sigma_i} = u_i \quad (i = 1, 2). \tag{1.1}$$

The necessary condition for the solvability of the inner Neumann problem (1.1) is the conservation of volume [5]

$$(1, \mathbf{u}) = \int_{\Sigma_1} u_1 d\Sigma_1 + \int_{\Sigma_2} u_2 d\Sigma_2 = 0. \tag{1.2}$$

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